Pipe networks - Handouts

Lecture 4. Pipe networks: branched systems and loops

1 Branched Pipes

- Branched pipes are more complex. Here we define the concept of a *node* – a point in a network where pipes meet.
- This leads to the consideration of continuity or conservation of mass – the flows into a node must be balanced by flows out of a node, since flow cannot accumulate (or diminish) there.
- We also adopt a *sign convention*, in this case that flows *into* the node are positive and flows *out of* the node are negative.
- Rigorous application of sign conventions is one of the keys to correctly solving these types of problems.

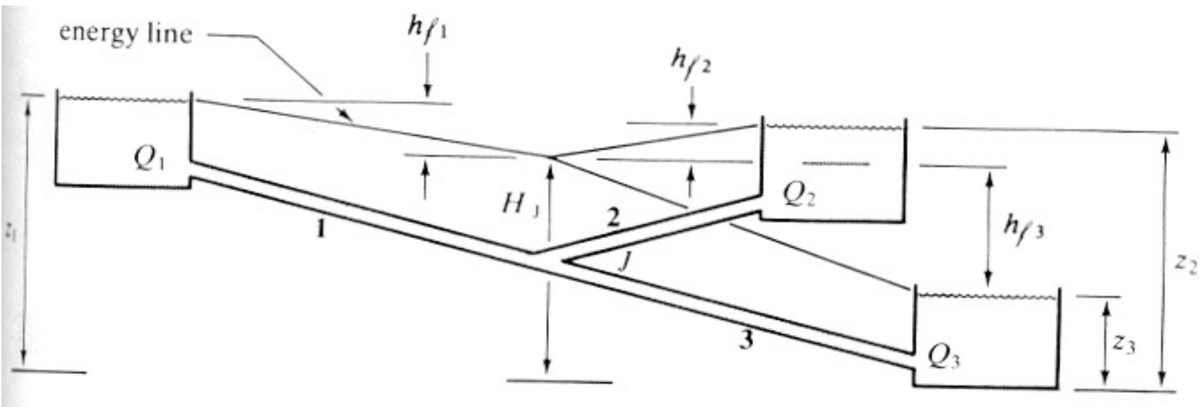

Figure 4.1

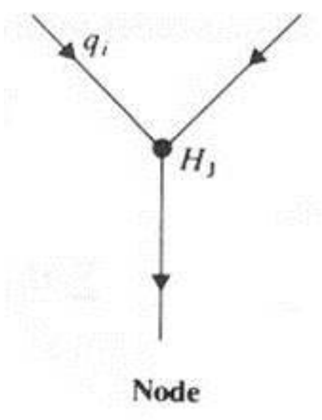

Figure 4.2

- For branched pipes, one can estimate a value of head at the junction, thus allowing the head loss in each of the pipes to be calculated, and then the flow. The flows at the junction need to sum to zero. If they do not, revise the estimate of height at the junction and repeat. This iterative processes can however be replaced by using the continuity equation at the node (with an assigned sign convention!) as 4th equation to constrain the algebraic problem (see class example);
- For design purposes, one can solve the problem by adopting an economic criterion to constrain the solution

Economic criterion to solve the "three reservoirs" problem

Let us consider the design problem of the so called 'three reservoirs's problem' (Figure 4.5). The aim is to design, i.e. calculate the best diameter set D_1, D_2, D_3 to convoy the assumed flow rates Q_1, Q_2, Q_3, with the established head losses.

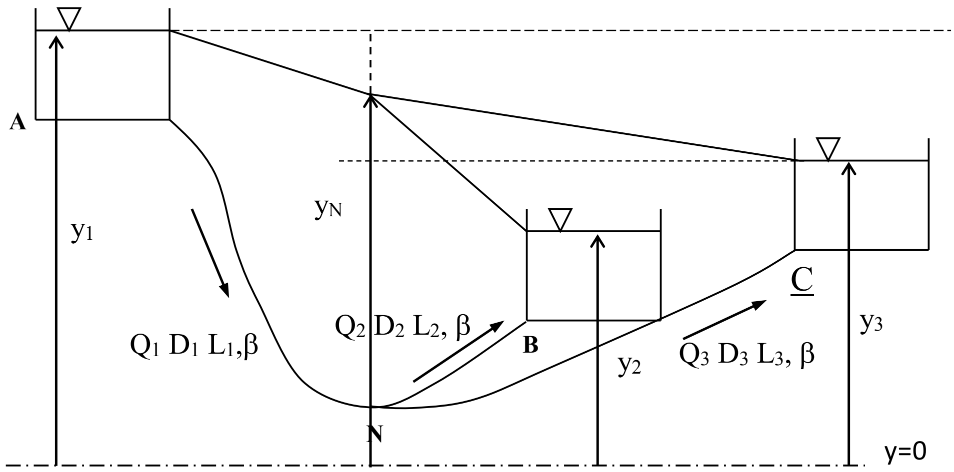

Figure 4.5

The piezometrics of the system can be infinite since is not known a priori the value of the piezometric head in the node N, i.e. we don't know the value of y_N. The equation of motion are

$$\begin{cases} y_1 - y_N = \beta \dfrac{Q_1^2}{D_1^n} L_1 \\ y_N - y_2 = \beta \dfrac{Q_2^2}{D_2^n} L_2 \\ y_N - y_3 = \beta \dfrac{Q_3^2}{D_3^n} L_3 \end{cases},$$

NOTE: the problem is undetermined from an hydraulic point of view and has infinity solutions in that it has four unknowns (D_1, D_2, D_3, y_N) and only three equations.

Furthermore, the continuity equation $Q_1=Q_2+Q_3$ in this case is NOT useful anymore since it declares just an identity.

To solve the problem we need to appeal to a criterion of economy in order to design the pipelines with the minor cost. This principle provides the fourth equation that makes determined the algebraic problem and therefore produces a unique solution.

The cost of a pipe can usually be expressed as

$$C = cD^{\alpha} L ,$$

being cD^{α} the cost per unit length which depends on the diameter D through the coefficient α>1 in order to account for the thickness augment with the diameter.
The total cost of the plant is therefore

$$C_t=C_1+C_2+C_3.$$

The three equations of motion can be expressed as a function of the unknown diameter

$$D_i = \sqrt[n]{\frac{\beta Q_i^2 L_i}{y_1 - y_N}} ,$$

that if inserted in the cost equation gives an equation which is a function of only the variable y_N. For sake of brevity we omit here this expression, however we explain in the following the reason of the presence of a minimum in such a cost function (Figure 4.6).

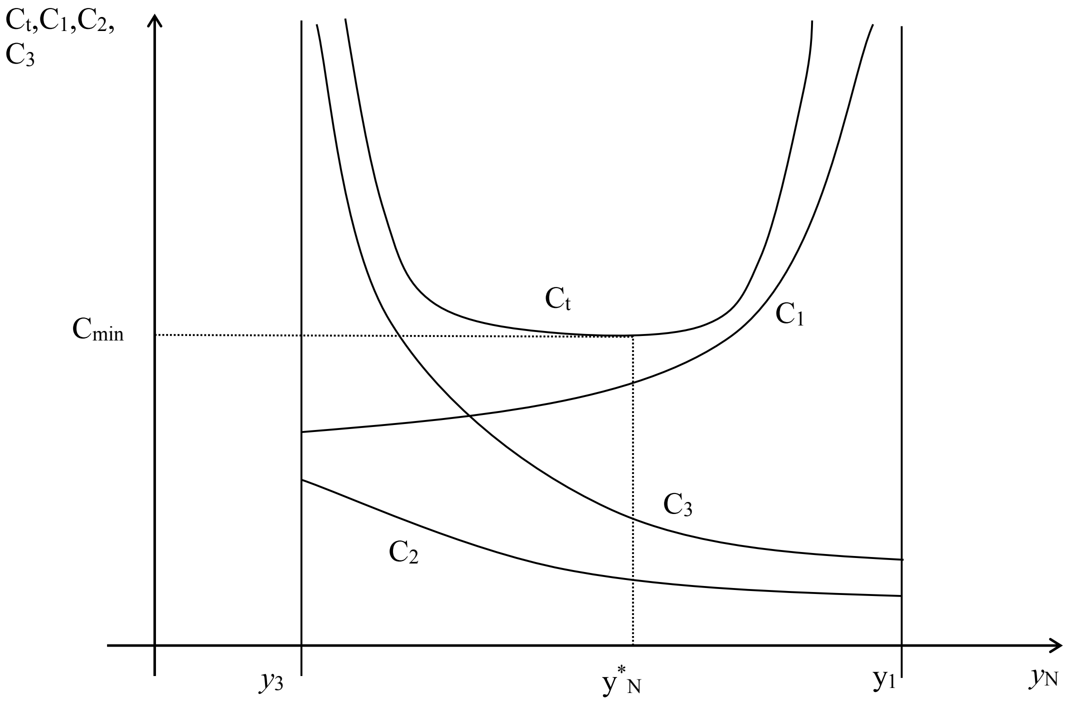

Figure 4.6

The minimum can be calculated imposing that the first derivative of the cost function has to be zero. The result expresses the so called 'minimum cost equation', which read

$$\boxed{\sum_i {}_{in} \frac{D_i^{\alpha+n}}{Q_i^2} = \sum_j {}_{out} \frac{D_j^{\alpha+n}}{Q_j^2}}$$

This equation has to be added to the three equations of motion in order to individuate the solution corresponding to the set of diameters which guarantee the minimum construction cost.

Design of Open branched networks

Let us have a look now at the scheme of Figure 4.8

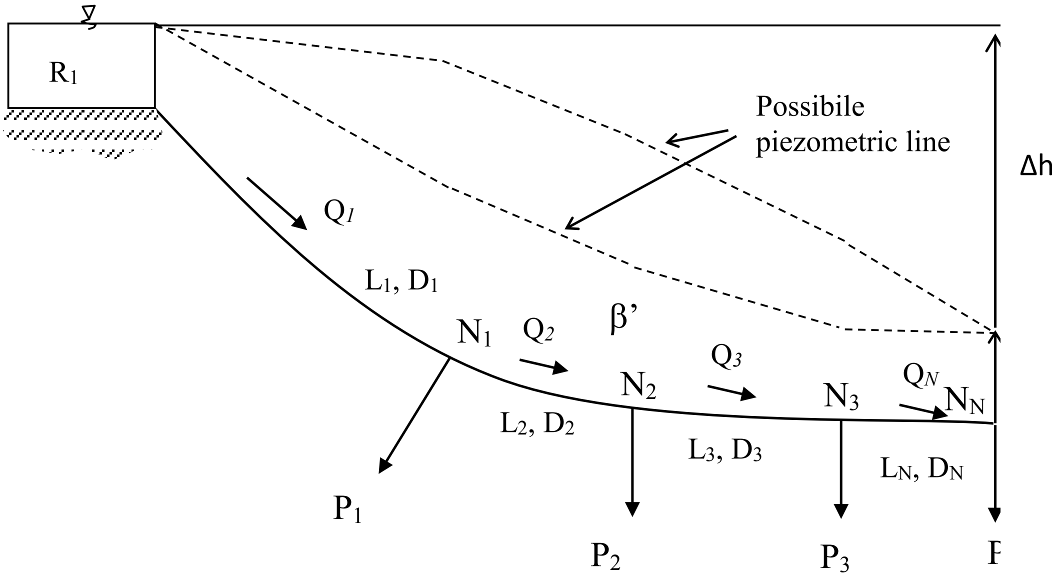

Figure 4.8

<u>Aim</u>: determining the best set of diameters of the main pipelines ABCDE the requested flow rates with the established piezometric head in the last nc

<u>Hypothesis</u>: neglect the cost of the secondary branches at each node.

The equation of continuity has to be respected in each node, for example a N_1 one has

$$Q_1=Q_2+P_1,$$

and so on.

This is an undetermined problem in that the piezometric of the system is r we must therefore appeal again at the condition expressing the minimum co cost.

For the node N_1 this condition reads

$$\frac{D_1^{\alpha+n}}{Q_1^2} = \frac{D_2^{\alpha+n}}{Q_2^2} \,,$$

and at the second node

$$\frac{D_2^{\alpha+n}}{Q_2^2} = \frac{D_3^{\alpha+n}}{Q_3^2} \,,$$

in general at the node N will be

$$\frac{D_{N-1}^{\alpha+n}}{Q_{N-1}^2} = \frac{D_N^{\alpha+n}}{Q_N^2} \,.$$

This transitive property leads therefore to the following relation

$$\frac{D_1^{\alpha+n}}{Q_1^2} = \frac{D_2^{\alpha+n}}{Q_2^2} = \frac{D_3^{\alpha+n}}{Q_3^2} = \ldots = \frac{D_N^{\alpha+n}}{Q_N^2} = A \,,$$

with A being a constant. By virtue of this relation and assuming for sake of convenience that n=5 and α=1, the generic diameter D_i becomes

$$\boxed{D_i = a\sqrt[3]{Q_i}},$$

where $a = \sqrt[6]{A}$.

In order to proceed we observe now that the total head loss has to be equal to the sum of the single head losses over each reach of length L_i and diameter D_i

$$\Delta h = \Delta h_1 + \Delta h_2 + \Delta h_3 + ... + \Delta h_N ,$$

i.e.

$$\Delta h = \beta \frac{Q_1^2}{D_1^5} L_1 + \beta \frac{Q_2^2}{D_2^5} L_2 + \beta \frac{Q_3^2}{D_3^5} L_3 + ... + \beta \frac{Q_N^2}{D_N^5} L_N .$$

Supposing now to neglect the little difference in the coefficients of roughness β, even if they should be *a rigori* a function of the unknown diameters and introducing the relation obtained for the diameters, we have

$$\Delta h = kQ_1^{1/3} L_1 + kQ_2^{1/3} L_2 + kQ_3^{1/3} L_3 + ... + kQ_N^{1/3} L_N$$

with $k = \frac{\beta}{a^5}$ =constant.

The previous relation allows to calculate the coefficient k as a function of known quantities

$$\boxed{k = \frac{\Delta h}{\sum_i Q_i^{1/3} L_i}} .$$

This very important result allows one to calculate the single head losses corresponding to the condition of max economy, i.e minimum construction cost, over each reach. <u>It has to be stressed the fact that in doing this we still do not known the best set of diameters that realize this condition.</u>

After having calculated the single head losses over each reach one can now calculate the corresponding diameter that realizes that situation

$$D_i = \sqrt[5.33]{\frac{\beta' Q_i^2 L_i}{\Delta h_i}}.$$

2 Loops

- In fact (as we have already seen in the Powerpoint shown in class) real supply systems involve loops.
- This presents a problem for analysis.
- This is of course generally done by computer (see for example the range of products at http://www.innovyze.com/).
- Some years ago there was a package called WATNET for water distribution networks which was ultimately subsumed into commercial products. Now however there is another WATNET here https://sites.google.com/site/drissamnouiri/watnet-software which I have not tested but would be worth a look. It appears to be free.
- We need to understand the basis for computational solutions.
- These tend to be based on the *Loop* or *Head Balance* Method.
- The basic principle is that in a loop such as that shown in the figure below, the sum of the head losses around the loop must be zero.
- This stands to reason as if you follow a flow around a loop, you cannot have a 'step' in head at any node.

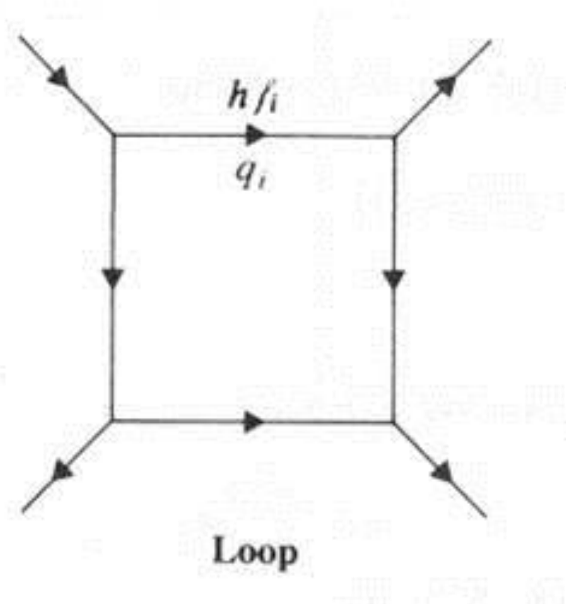

Figure 1

- We thus need to satisfy

$$\sum_{i=1}^{m} h_{fi} = 0$$

Equation 1

- Around m pipes.
- We adopt another sign convention, which is usually *flows clockwise around a loop are positive*.
- Failure to apply the sign convention rigorously is the most common source of student error in solving this sort of problem!

3 The Hardy-Cross Method

- The best known method for solving Equation 1 derives from Hardy Cross in 1936. Hardy Cross was a professor at University of Illinois at Urbana-Chamaign and also invented the moment distribution method for structural analysis, which has a certain amount in common with his method for pipe loops.
- This method depends on knowing the inflows and outflows for a loop, and determines the values of head at intermediate nodes and the flow in each pipe.
- It works as follows:
  1. assume values of q_i to satisfy $\sum q_i = 0$.
  2. calculate h_{fi} from q_i (using any suitable method for calculating head loss, usually λ).
  3. If $\sum h_{fi} = 0$, then the solution is correct.
  4. If $\sum h_{fi} \neq 0$, then apply a correction to q_i factor and return to step 2
- The trick is of course to calculate the correction factor to apply, which is done as follows.

4 Hardy Cross Correction Factor

- Since:

$$h_f = \frac{fLu^2}{2gD}$$

- We can say for a given pipe:

$$h_f = kQ^2$$

- Where k is a constant.
- If we esstimate a flow in pipe i inaccurately as q_i, we can say that the true flow is:

$$Q = (q_i + \delta q)$$

- We can calculate a head loss from our calculated flow:

$$h_{fi} = kq_i^2$$

- But the true head loss is then:

$$H_{fi} = k(q_i + \delta q)^2$$

- We can then expand this according to the Binomial Theorem:

$$H_{fi} = kq_i^2 \left[1 + 2\frac{\delta q}{q_i} + \frac{2(2-1)}{2!}\left(\frac{\delta q}{q_i}\right)^2 + \cdots\right]$$

- Ignoring terms of second order or higher, which will be very small for $\delta q_i << q_i$, we have:

$$H_{fi} = kq_i^2 \left[1 + 2\frac{\delta q}{q_i}\right]$$

- Also, for a loop, we must satisfy:

$$\sum H_{fi} = 0 = \sum kQ_i^2 + 2\delta q \sum k\frac{q_i^2}{q_i}$$

$$\Leftrightarrow \sum h_{fi} + 2\delta q \sum \frac{h_{fi}^2}{q_i}$$

$$\Leftrightarrow \delta q = -\frac{\sum h_{fi}}{2\sum \frac{h_{fi}}{q_i}}$$

5 Example

- The Hardy Cross method is best explained by way of examples.

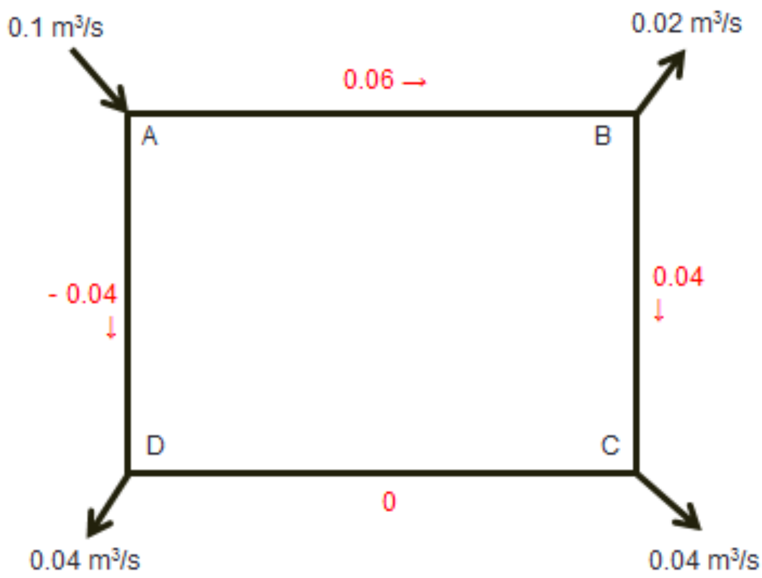

Figure 2

- The figure above shows a simple loop with stated (in black) the known inflow at point A and the outflows (water demand) at points B to D.

- In this example all the pipes are 1000 m long. AB and DA have a diameter of 150 mm whilst BC and CD have a diameter of 125 mm. All have a k_s of 0.03 mm.
- The first stage of the calculation is to assume flows in the pipes AB, BC, CD and DE.
- It does not matter what flows are assumed provided they are consistent with the known inflows and outflows.
- Also, the sign convention (clockwise flows are positive) must be observed.
- The initial estimate for this example is shown in red on the figure.

  We set up a tabulation of the results and calculate the flow velocities from the estimated flows and pipe diameters:

Pipe	q_i (m^3/s)	L (m)	D (m)	u (m/s)	h_{fi} (m)	h_{fi}/q_i
AB	0.06	1000	0.150	3.40		
BC	0.04	1000	0.125	3.26		
CD	0	1000	0.125	0		
DA	-0.04	1000	0.150	2.26		

- The next stage is to calculate the head loss for each pipe based on u, L and D, using any suitable method such as finding λ.
- h_{fi} goes in the penultimate column.
- Note that if the flow is negative in the sign convention, the head loss is negative (i.e. it is a head gain!) too.
- Note also that the numbers in this example are not especially realistic – the head losses are massive.

Pipe	q_i (m^3/s)	L (m)	D (m)	u (m/s)	h_{fi} (m)	h_{fi}/q_i
AB	0.06	1000	0.150	3.40	59.58	
BC	0.04	1000	0.125	3.26	68.32	
CD	0	1000	0.125	0	0	
DA	-0.04	1000	0.150	2.26	-28.14	

- Then the calculated head loss is divided by the flow estimate from the second column to give the number in the final column (enter zero if the flow is zero, bearing in mind that in a spreadsheet calculation, which is basically a good way to do this, there will be a divide by zero error you will have to trap):

Pipe	q_i (m^3/s)	L (m)	D (m)	u (m/s)	h_{fi} (m)	h_{fi}/q_i
AB	0.06	1000	0.150	3.40	59.58	992.97
BC	0.04	1000	0.125	3.26	68.32	1707.94
CD	0	1000	0.125	0	0	0
DA	-0.04	1000	0.150	2.26	-28.14	703.49

This is given by this divided by this

Note the minus signs cancel in the division

- We now work out the correction factor which involves summing each of the last two columns:

Pipe	q_i (m^3/s)	L (m)	D (m)	u (m/s)	h_{fi} (m)	h_{fi}/q_i
AB	0.06	1000	0.150	3.40	59.58	992.97
BC	0.04	1000	0.125	3.26	68.32	1707.94
CD	0	1000	0.125	0	0	0
DA	-0.04	1000	0.150	2.26	-28.14	703.49
				Σ =	99.76	3404.40

- The two sums go into the correction factor calculation:

$$\delta q = -\frac{\sum h_{fi}}{2\sum \frac{h_{fi}}{q_i}} = -\frac{99.76}{2 \times 3404.40} = -0.015$$

- The correction factor is then *added* to the original estimate of flow for *every* pipe in the system. For example the new flow in AB becomes 0.06 + -0.015 = 0.045, and the calculation re-worked:

Pipe	q_i (m^3/s)	L (m)	D (m)	u (m/s)	h_{fi} (m)	h_{fi}/q_i
AB	0.045	1000	0.150	2.57	35.49	782.69
BC	0.025	1000	0.125	2.07	29.38	1159.01
CD	-0.015	1000	0.125	1.19	-10.66	727.30
DA	-0.055	1000	0.150	3.09	-50.13	917.20
				Σ =	4.09	3586.20

- This time δq comes out as -0.001.
- The process continues until the sum of the penultimate column, Σh_{fi} is close to zero and the correction factor is negligibly small. Usually three or four iterations are sufficient to achieve this in simple loops – in fact the second iteration just above is pretty close already.

6 Complex Loops and Example

- More complex loops can still be solved in this way, for example that in the figure below. The inflows, outflows and initial estimates of the flow in the pipes have already been added (though they are shown in litres/s here).

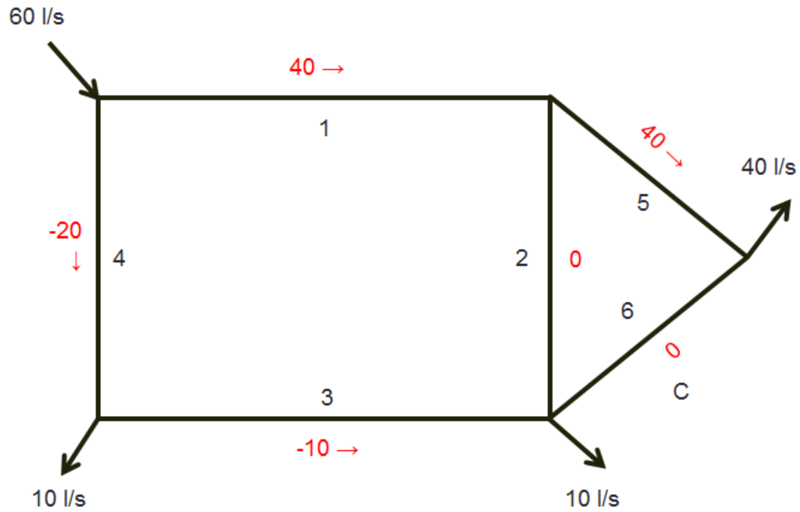

Figure 3

- The subtlety here is that pipe 2 appears in both loops, but with its flows having opposite signs: the left hand loop flow in pipe 2 is clockwise so positive, but in the right hand loop the same flow in this pipe has to be considered negative as it is anti-clockwise from the perspective of this loop.
- We set up a double tabulation, as shown below, but work the calculation in basically the same way.
- Consider all pipes to be 1000 m long, 100 mm in diameter and have a k_s of 0.03 mm.

Left Hand Loop				Right Hand Loop			
Pipe	q_i (litres/s)	h_{fi} (m)	h_{fi}/q_i	Pipe	q_i (litres/s)	h_{fi} (m)	h_{fi}/q_i
1	40	28.5	0.71	5	40	28.5	0.71
2	0	0	0	6	0	0	0
3	-10	-2.2	0.22	2	0	0	0
4	-20	-8	0.40				
	Σ=	18.3	1.33		Σ=	28.5	0.71

These two must always be equal and opposite – same pipe in the two loops, but opposite sign conventions

- It will hopefully be obvious that we now calculate *two* correction factors, one for each loop:
- δq (left hand) =-6.87 and δq (right hand) = -20
- We then apply each correction factor to the flow estimates for its own loop, so the new flow in pipe 1 is as follows:

40 + - 6.87 = 33.1 litres/s

- Whilst that for pipe 5 is:

40 + -20 = 20.0 litres/s

- BUT (and this is the tricky part) pipes that appear in both loops have both correction factors applied, taking into account the sign convention. So, the new flow for pipe 2 in the left hand loop becomes:

0 + -6.87 - -20 = 13.1 litres/s

- Whilst that for pipe 2 in the right hand loop is:

0 + -20 - -6.87 = -13.1 litres/s

- Which is of course still equal and opposite to its equivalent in the left hand loop where the sign convention dictates its flow has the opposite sign.
- The calculation continues as before until the sum of the head losses around the loops becomes small and the correction factors negligible.

7 Nodal or Quantity Balance Method

- The head balance method outlined above only works if the inflow and outflow quantities are known. If, as is quite possible, the heads at the inflow and outflow points are known, but not the flows, another approach is necessary. This is the quantity balance method.
- It is derived, and works in a similar way, but instead of heads being summed around a loop, the flows at each node are summed. If they are not zero, correction factors are calculated from:

$$\delta H = \frac{2\sum q_i}{\sum \frac{q_i}{h_{fi}}}$$

8 Example

- A good example of the nodal method is to re-work the previous branched pipes example using a tabular approach. This is done in class.
- For systems with many nodes, the iteration involved becomes complex as an answer must be converged at each node before moving to the next, and then once all nodes have been done the calculation must restart at the first node, until changes become insignificant. I have a spreadsheet example of such a calculation, but it is not sensible to do it by hand calculation.